CHAPTER 10

MORE THAN MERE HANDWAVING

Gesture and Embodiment in Expert Mathematical Proof

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For the budding mathematician, the ability to comprehend and produce rigorous proofs marks the transition from a novice student or consumer of mathematics to a more expert practitioner. By the time a student begins graduate school in mathematics, he or she is capable of both understanding and creating formal proofs of known mathematical results and is expected to generate new proofs and theorems. Often the focus is on the proofs themselves, the stable, shared outcomes of a process of proving.

Emerging Perspectives on Gesture and Embodiment, pages 227–246

In this chapter, by contrast, the focus is on proof as an active practice carried out by knowledgeable mathematicians, looking specifically at the relationship among gesture, speech, thought, and technical definitions. This case study is part of a larger ongoing research project on expert proof, in which we focus on how advanced students of mathematics conceptualize proof, how they collaborate to produce a proof, and how they coordinate multiple modalities during their practice, modalities that include speech, gesture, and inscriptions, both symbolic and graphic.[1] Our general thesis is that these resources are deployed in a complex but coherent manner to construe and scaffold the building of the desired intellectual product, the mathematical proof.

The goals of this chapter are twofold. First, we argue for the importance of gesture in mathematical practice, including proof, and review recent research that places the body at the center of mathematical practice. Second, we report on a case study of mathematical proving that treats proof as a human activity, extended over time and performed in a social and material context (Edwards, 2010; Marghetis & Núñez, 2013). Specifically, we document the various kinds of gestures produced by mathematical experts while they elaborate a proof and argue that these spontaneous gestures suggest that experts' reasoning about mathematical concepts is fundamentally metaphorical and embodied. By examining the practice of proving from a multimodal perspective, we hope to shed light on the nature of proof itself, as a product but also as a human practice.

FROM PRODUCTS TO PRACTICE IN THE STUDY OF MATHEMATICS

Traditionally, foundationalist accounts of mathematics (e.g., Benacerraf & Putnam, 1983) have focused exclusively on the products of mathematical activity—proofs and theorems—while ignoring the rich practices that contribute to, and potentially constitute, the core of mathematics. This tendency to analyze static, abstract products rather than dynamic, human practices is reminiscent of early formalist accounts of natural language, where, as Kendon (2008) notes, "what is transferred to paper is abstracted away from what is actually done within an enacted utterance" (p. 357).

Over the last few decades, researchers in mathematics education have broadened their focus beyond proof as a product to investigate proving as a cognitive *process* and a form of regimented discourse (e.g., Balacheff, 1991; Boero, 2007; Hanna, 1991; Knuth, 2002), although this literature has generally focused on secondary school mathematics (typically geometry), rather than on more advanced levels. Only recently, however, have these investigations attended to modalities beyond speech and inscriptions. But without a broader scope that includes the mathematician's body, essential aspects of communicative practice are invisible (Goldin-Meadow, 2003; Goodwin, 2000; Kendon, 2004, 2008; McNeill, 1992, 2000). Indeed, the particular form of kinesis represented by hand and arm gestures has unique communicative potential during situated proving. Manual gestures are often enacted unconsciously and can convey information that might be consistent with, complementary to, or different from the content of accompanying speech (Kita, 2000; McNeill, 1992) or inscriptions (Marghetis & Núñez, 2013). In fact, we would go further to propose that gesture and other bodily movement is essential not only in communicating about mathematics but also in the intellectual construction of mathematics. In short, the mathematician's body may be a constitutive part of his or her situated proving.

This claim requires empirical support, of course, and later in this chapter we describe a recent study that supplies such evidence. But before turning to the case study, we briefly review some of the recent literature on co-speech gesture and mathematics.

GESTURE: MEANINGFUL MOVEMENTS OF THE BODY

Gesture is universal, typically unmonitored by the speaker, and essential to communication (Goldin-Meadow, 2003; Kendon, 2004; McNeill, 1992). Most important, gesture offers a "window into the mind" (Goldin-Meadow, 2003, p. 1), in that gestures can provide clues to how a person is conceptualizing the situation being discussed. When co-produced with abstract thinking, a speaker's gestures may parallel the metaphorical reasoning they exhibit in speech (Cienki, 1998; McNeill, 1992, 2000; Núñez, 2006), and give us insight into their representation of mathematical concepts and solution strategies (e.g., Alibali, Bassok, Solomon, Syc, & Goldin-Meadow, 1999; Garber & Goldin-Meadow, 2002). These meaningful movements of the hands and body, moreover, are meaningful not only for the speaker but also for the listener: "perceiving hand movements during speech modulates the distributed pattern of neural activation involved in both biological motion perception and discourse comprehension, suggesting listeners attempt to find meaning, not only in the words speakers produce, but also in the hand movements that accompany speech" (Dick, Goldin-Meadow, Hasson, Skipper, & Small, 2009, p. 3509). And the influence of gesture is not limited to concrete, literal language: Evidence from electrophysiology suggests that the cross-modal coordination of speech and gesture shapes the neural response to metaphorical language (Cornejo et al., 2009). Because speech and gesture differ in their communicative possibilities, their combination may "package" complementary forms of information within the same discourse: linear, symbolic verbal language, on the one hand, and global, instantaneous imagery, on the other (Goldin-Meadow, 2003; Kita, 2000; McNeill, 2000).

Various means of categorizing or characterizing gestures have been developed. In this study, an initial distinction is made between gestures whose purpose is to facilitate the interaction between the speaker and interlocutor and gestures that refer to the content of the conversation. Bavelas and her colleagues have called these two types of gestures "interactive" and "topical," respectively (Bavelas, Chovil, Lawrie, & Wade, 1992). The analysis elaborated here focuses on these "topical" (also called "representational") gestures, especially the way they can embody abstract mathematical concepts during proving. McNeill (2000) has further characterized gestures along four dimensions: iconicity (resemblance to concrete referents), metaphoricity (reference to abstract entities), deixis (pointing or context-dependence), and "temporal highlighting" or beats (giving emphasis through repetition, i.e., producing "beats" in time with speech). Kendon (2004) makes even further distinctions, discussing three varieties of gestural representation that are distinguished by the relationship of the gesture to its referent: enactment, depiction, and modeling. In enactment, the motor action is meant to reproduce some features of the activity being represented. For example, gesturing in the air as if you were drawing a graph would be an enactment of the physical act of creating a graph. In depiction, the gesture "creates an object in the air" (p. 160), typified by using the index finger to trace an object's shape. In the example of graphing, while enactment would represent the action, a finger tracing a graph in the air would be a representation of the finished product (in this case, the act of drawing and the actual graph that is produced may be manifested in similar gestures). In modeling, the gesturing body part stands in for another object, as when a fist represents a stone, or a speaker indicates a tangent line by holding his or her palm at an angle. As we shall see, these distinctions will be useful for describing and interpreting the gestures generated by doctoral students while they work together to find a proof.

MATHEMATICS AND GESTURE

Gesture is prevalent in mathematical discourse as in other kinds of speech (McNeill, 1992). Previous research on gesture and mathematics has looked at various mathematical tasks, from simple (e.g., learning to count; Alibali & diRusso, 1999) to advanced (e.g., discussing differential equations; Rasmussen, Stephan, & Allen, 2004). Gesture can offer insight into how students think while solving problems and into the effectiveness of teachers' communication (Alibali et al., 1999; Edwards, 2009; Garber & Goldin-Meadow, 2002; Goldin-Meadow, Cook, & Mitchell, 2009; Goldin-Meadow, Kim, & Singer, 1999; Goldin-Meadow, Nusbaum, Kelly, & Wagner, 2001; Goldin-Meadow & Wagner, 2005). In a study of gestures used during algebra problem solving (Alibali et al., 1999), for instance, researchers found a relationship between the form of representational gestures and the semantic content of the mathematics problem. If a problem was stated in a way that involved step-by-step, discrete changes, the gestures that participants used to describe that problem also tended to be “discrete” (e.g., a sequence of taps or beats). Conversely, a problem phrased in terms of continuous change elicited sweeping, arcing, or other “continuous” gestures. Gesture type also tended to reflect the particular kind of solution attempted (one type representing a view of the problem as continuous and the other as discrete). However, when the participant’s speech and gesture did not match, they tended to use the strategy associated with their gestures rather than that associated with their speech. As the authors noted, “Our findings suggest that mental representations often include visual or perceptual information, which may at times be more readily expressed in gesture than in speech. In these cases, spontaneous gestures can be a valuable tool for illuminating mental representations” (Alibali et al., 1999, p. 332).

While early investigations into mathematics as an embodied activity generally focused on mathematical language (Fauconnier & Turner, 2002; Lakoff & Núñez, 2000; Núñez, 2006), typically taken from textbooks and conventional language, more recently mathematical gesture has provided converging evidence for the role of embodied, possibly unconscious, cognitive processes like conceptual metaphor (Lakoff & Johnson, 1980) and fictive motion (Talmy, 2000). Spontaneous gesture has provided evidence that arithmetic, for instance, is conceptualized using metaphors of Object Collection or Motion Along a Path (Lakoff & Núñez, 2000; Núñez & Marghetis, in press). Analyses of gestures for fractions, geometry, and graphing have revealed that implicit, embodied conceptual mappings are manifested in bodily action (e.g., Edwards, 2009; Font, Bolite, & Acevedo, 2010). Even mathematicians’ gestures about highly technical mathematical concepts reveal such embodied thought, at least when produced in a pedagogical context (Núñez, 2006, 2008).

Conclusions drawn from classroom contexts and mathematically naïve undergraduates, however, may not extend to the non-pedagogical activities of expert mathematicians. Here we elaborate on a study that utilized video and audio data recorded during sessions in which doctoral students in mathematics worked together to create a mathematical proof for a conjecture that was new to them. In Marghetis and Núñez (2013), we applied the tools of gesture studies, cognitive linguistics, and embodied cognition to examine the spontaneous co-speech gestures of doctoral students as a means of better understanding their thinking about a proof-in-progress (see also Edwards, 2010). In contrast to many studies of gesture and mathematics, this setting was not pedagogical. Instead, the participants in this study were asked to do something that they had done many times before—create a valid proof—but because the conjecture was new to them, this task nevertheless represented an intellectual challenge and offered a context for sophisticated mathematical discourse. This approach offers an opportunity to examine proving as a central practice of mathematicians. The participants' gestures thus accompanied authentic mathematical practice and have the potential to reveal textured aspects of the participants' mathematical thought.

A CASE STUDY OF EXPERT PROOF

When mathematicians work on a proof, they talk, scribble, on blackboards and in notebooks; stare pensively into empty space; and, crucially, gesture even when working alone. These co-thought and co-speech gestures might be mere "handwaving," empty accompaniments to "core" processes of logical inference. As we have seen, however, there is mounting evidence that gestures are central to, and revealing of, abstract thought more generally. Might expert mathematical discourse reveal a relationship between gesture form and the inferential structure of the mathematical argument?

To answer this question, we went to a natural site of mathematical proving: the academic office. We arranged for twelve doctoral mathematics students from a large American research university to collaborate on a non-trivial proof, working in pairs at a blackboard. They had been enrolled in the graduate program for various lengths of time, from ten months to three years, and were working in a variety of subfields within mathematics. During a 90-minute session, the participants were first interviewed about their mathematical specializations, as well as their approaches to learning and teaching proof. After the interview, the pairs of students were asked to prove the conjecture below, provided to them on a sheet of paper:

> Let f be a strictly increasing function from [0, 1] to [0, 1].
> Prove that there exists a number a in the interval [0, 1]
> such that $f(a) = a$.

This problem was selected, in part, because the mathematician who proposed it reported that when proving the theorem, he experienced a palpable sense of motion. We also checked with faculty at the university to determine that the theorem was not presented in any of the graduate analysis courses, and thus was likely to be new to the students. Although the problem comes from real analysis, it does not require specialized or advanced knowledge within that subdiscipline. The proving of this theorem, therefore, presented an excellent opportunity to challenge and examine the mathematical thinking of the participants.

Participants were given up to 40 minutes to solve the problem, working in a room with a blackboard, on which they were asked to record their work. The entire session was videotaped, but the investigator (the second author) was not present while the students worked on the proof. Once the pair was satisfied with the proof, or the 40 minutes had passed, the investigator returned to the room, and the participants were asked to explain their proof or their progress to that point.

Here we focus on an analysis by Marghetis and Núñez (2013) of the mathematical speech and gestures from these video-recorded sessions. The analysis concentrated on gestures used to represent mathematical content, that is, gestures that were co-produced with talk of particular mathematical concepts. Specifically, Marghetis and Núñez examined representational gestures produced in conjunction with two categories of conceptual content: static and dynamic mathematical ideas.

By static ideas, we mean those mathematical concepts that are hypothesized to rely on static spatial schemas such as containment—ideas like mathematical sets, which can be conceptualized metaphorically as containers for their elements. By contrast, dynamic ideas are mathematical concepts that are thought to rely for their conceptualization on intuitions of space and motion. This distinction between static and dynamic ideas grew out of work by Talmy (2000), who noted that language typically used to express motion is often used in the absence of literal motion, a phenomenon known as fictive motion. For instance, we say that "the road <u>runs through</u> the hills of Tuscany," even though roads are static and thus incapable of literally running anywhere. Lakoff and Núñez (2000) noted that a similar phenomenon exists for technical mathematical concepts—including the notions of functions, continuity, and limit—where the technical definitions are entirely static and yet mathematicians reliably talk about such concepts using dynamic language.

For illustration, consider the concept of a limit, a central notion in calculus. Technically, the limit of a function is defined by a chain of inequalities:

Let a function f be defined on an open interval containing a,
except possibly at a itself, and let L be a real number.
Then $\lim(x \to a)\ f(x) = L$ means that, for all $\varepsilon > 0$, there exists $\delta > 0$,
such that whenever $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Note that the limit notation includes a small arrow, which might suggest that the definition of a limit of a function would include some notion of dynamism. The technical definition of a limit, however, refers only to static universal and existential quantifiers, static numbers, motionless arithmetic differences, and static inequalities. Nowhere in this definition is there any mention of movement. Mathematicians, in contrast, speak of a function "tending to," "moving toward," or "reaching" a limit—all of which, unlike the formal definition, invoke a sense of motion (Núñez, 2006). These expressions are all instances of fictive motion, invoking a trajector (moving actor) in motion across a landscape (non-moving background)—despite the fact that there is no literal motion involved. Similar instances of fictive motion inject dynamism into a wide range of statically defined mathematical entities. A function, for example, is formally defined as a static relation between two sets, the domain and the range, but mathematicians nevertheless routinely describe functions dynamically as "*reaching* an asymptote," "*going down* towards a minimum," or "*oscillating*," in each case evoking a construal in which an imaginary trajector travels along the path of the function (Núñez, 2006). Fictive motion is similarly at work when we say that sequences are "approaching," "decreasing," or "converging," and when arithmetic is construed as motion along a number line. Specific forms of dynamism, therefore, are present throughout the language of mathematics, showing up in the discourse surrounding continuity, functions, and even arithmetic, which lends credence to the claim that mathematical thought builds on dynamic, metaphorical, and embodied construals.

While previous analyses have argued that these ideas rely in varying ways on spatial intuitions like motion or containment (Lakoff & Núñez, 2000), this remained largely speculative: There was no empirical evidence that dynamic, spatial intuitions played a role in real-time, expert practice. Language, after all, can mislead, and the archaeology of language is a fraught enterprise: "*fast*ing," for instance, has nothing to do with speed or motion. It's entirely possible that the rampant fictive motion in conventional mathematical language is similarly misleading, suggesting dynamism where there is none. As we have pointed out, however, mathematical discourse is not limited to speech but also involves gesture. Can gestures, meaningful movements of the body, provide additional evidence for dynamism in mathematical thought?

EVIDENCE FOR FICTIVE MOTION IN GESTURE

To zoom in on this mathematical content of interest, Marghetis and Núñez (2013) generated a list of lexical items thought to be associated with fictive motion or dynamic conceptual metaphors, drawing on previous theoretical work within cognitive linguistics (Talmy, 2000). These included mathematical terms (e.g., function, continuity, limit, contain), verbs of motion (e.g., to cross, to move, to jump), and spatial terms (e.g., up, between, left).

Mathematical terms referring to static concepts (e.g., set, point) were also identified. Of course, many everyday concepts are open to multiple construals, and technical mathematical concepts are no different. The notion of continuity, for instance, can be construed dynamically as movement without jumps—but also as "Preservation of Closeness," a static construal (Núñez, Edwards, & Matos, 1999; Núñez & Lakoff, 1998). As we shall see, this was reflected in participants' embodied discourse, where a single expression was produced on multiple occasions with gestures that varied in dynamicity.

Gestures co-produced with these targeted lexical items were then classified as dynamic, static, or ambiguous.[2] Because gestures are co-speech motor actions, they necessarily involve dynamic movement. To differentiate between gestures for which dynamism was an artifact of gesture production, and those gestures for which dynamism was truly expressive of mathematical thinking, we devised a coding scheme that attended to details of the motion and timing of the co-speech gesture. A gesture was coded as: (a) dynamic if it used smooth, unbroken motions; (b) static if it consisted of staccato strokes and segmented motions, or of a smooth motion bookended by staccato gestures; and (c) ambiguous if it did not fit into either of these categories.

The graduate students in our study were highly skilled, and thus it was no surprise that every dyad arrived at a reasonable solution that included most of the elements of a complete proof. Half of the dyads even finished early, before the end of the allotted time. In all cases, participants' proofs followed the same outline as the proof supplied by the mathematician who proposed the problem. All six dyads made extensive use of diagrams—primarily graphs of "generic" increasing functions, or sections of functions—in combination with extensive symbolic inscriptions. However, the sequence in which the various modalities were used, including speech, symbolic and graphical inscriptions, gesture and gaze, varied a great deal within and among the pairs of participants. Some participants spent several minutes looking at the paper on which the conjecture was printed or staring into space, others immediately sketched a graph on the blackboard, while several initially discussed what they knew or could immediately conclude about the conjecture (see Edwards & Harel, 2009). These discussions were always accompanied by gestures, some of which were used pragmatically to manage the interaction (turn-taking, emphasis, etc.) while others occurred in conjunction with the use of content-rich mathematical speech.

Examples of Co-speech Representational Gestures

As noted, gestures used to represent the conceptual content of the mathematical discourse were the focus of our analysis. To give a flavor of the range of these representational gestures, below are several examples of dynamic and static representational gestures co-produced with similarly coded lexical items.

Dynamic Gestures

The technical concept of “increase” offers an example of a mathematical construct that seemed to demand an exclusively dynamic treatment in gesture. Participants produced fourteen representational gestures that co-occurred with the lexical items “increasing” or “increase.” All of the co-produced gestures were dynamic.

In Figure 10.1, the participant is saying “increasing sequences” and while doing so produces a smooth, unbroken motion, co-timed with his speech. At the onset of the word *increasing*, he begins to fluidly move his left hand upward and toward the right, with his thumb pointing in the direction of motion (Figure 10.1A). As he reaches the end of the word, the motion of his left hand slows slightly while his right hand begins to accelerate, once again moving upward and to the right in the direction of his extended thumb. As he begins to say “sequence,” his right hand reaches its top speed (Figure 10.1B). Both hands begin to slow to a stop (Figure 10.1C), and their retraction is co-timed with the end of the word “sequences.” Neither his gaze nor his thumbs are directed toward the blackboard or any inscription on it; rather, the deictic aspect of the gesture is used to indicate the direction that the graph of any increasing function would take (up and to the right).

In Figure 10.2, the student is at the blackboard writing a series of inequalities that contradict the assumption that the function is increasing. As he finishes writing, he steps back from the blackboard, drops his hands to his side, and states, “So that contradicts uhhh increasing” (Figure 10.2A). Precisely co-timed with the onset of the word *increasing*, his right hand flies upward and to the right, with his index finger extended in a prototypical pointing hand shape (Figure 10.2B). As he finishes saying “increasing,” his right hand slows, pointing to the right of the blackboard for a moment

Figure 10.1 Dynamic deictic gesture while saying “increasing sequence.” Both hands evoke the sequence’s fictive motion.

Figure 10.2 Dynamic gesture while saying “increasing.” A fluid rightward hand movement is co-timed with speech.

(Figure 10.2C), before smoothly dropping to his side. This is a quintessential example of dynamic co-speech gesture, consisting of a continuous motion co-timed with a dynamic lexical item.

Static Gestures

Static gestures often accompanied discussions of closeness and containment. In Figure 10.3, the student is discussing the values of a function and wants to consider only those values contained within a restricted region. At first, all her gestures are deictic, anchoring her discussion to the graph

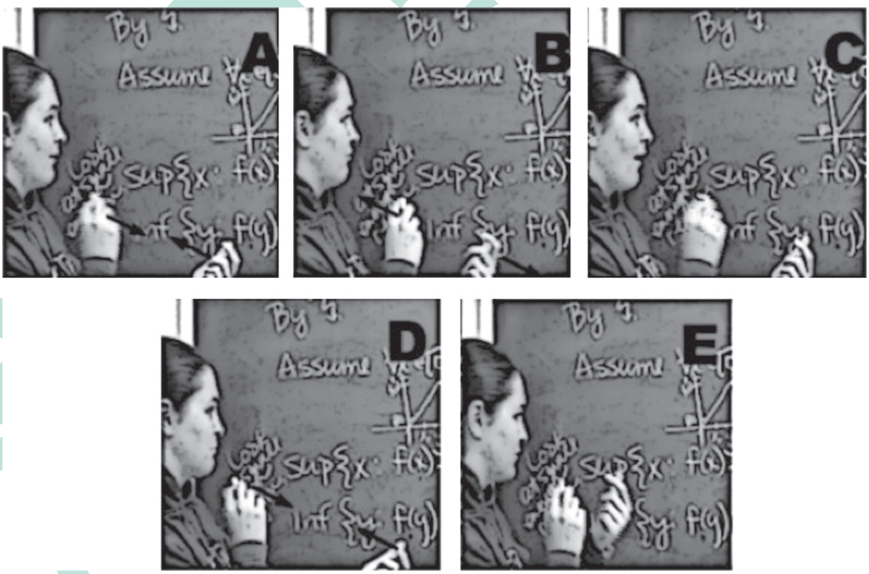

Figure 10.3 A static gesture co-timed with the utterance, “sort of, small enough.”

she has drawn on the blackboard. She introduces the region of interest by saying, "Well, if you look at a, sort of, small enough, sort of. . . ." When she begins to describe the region of interest ("Well . . ."), she retracts her hands from the blackboard and positions them in front of her chest, pointing toward each other, and pauses as she says, "sort of" (Figure 10.3A). Co-timed with the onset of the word *small*, she quickly moves her hands toward each other, stopping when they are 10 cm apart, and then quickly retracting her hands to their original distance (Figure 10.3A–C). Co-timed with the word *enough*, she repeats the same inward staccato stroke, stopping abruptly when her hands are at the same distance and retracting once again (Figure 10.3D–E). By indexing two particular points in space with a repeated beat, she evokes the endpoints of a delimited region containing the function's domain. This gesture was thus coded as static because it consists of distinct beats, indexing exact points in space.

Variable Gesture for the Same Utterance

Often a single utterance was amenable to different kinds of representation in gesture. The utterance "to the left," for instance, received both dynamic and static treatments in gesture (Figure 10.4).

In the second frame of Figure 10.4 (labeled B), the participant is producing a static gesture co-timed with the utterance "to the left." One participant suggests to his collaborator that they should look for "implicit continuity to the left of what we're talking about." As he says "to the left," he forms his thumb and forefinger into a U-shape, with the two digits representing the boundaries of a specific interval, performs one forward beat with this hand shape, and then holds his hand still for nearly a full second. This gestural representation of "to the left" captures the static notion of containment within a fixed region to the left of a reference point.

Figure 10.4 Two gestures co-produced with the utterance "to the left." While the gesture on the left is dynamic, the one on the right is static.

The first frame of Figure 10.4 (labeled A) shows the same lexical affiliate ("to the left") co-timed with a dynamic gesture. While discussing the intersection of a function and the line $y = x$, a participant asks whether the function was "going a little bit to the left." As he begins to say "to the left," he points his forefinger to the left and fluidly moves his hand in that direction, retracting his hand as he says "left." In contrast to the example of a fixed (static) region to the left, above, this gesture represents a dynamic sense of the fictive motion of the function as it "moves" in a particular direction.

These two cases illustrate that gestures may share a lexical affiliate but differ in their dynamism. Although highly synchronous with the utterance "to the left," these two gestures exhibit different kinetics and hand shapes.

VARIETIES OF METAPHORICAL GESTURES IN MATHEMATICS

Kendon's (2004) distinctions among types of gestures can help us understand the various ways in which participants used gesture to represent mathematical concepts. As noted above, during enactment the gesture enacts or reproduces some aspect of a physical action. The enactments produced by all of the participants were directly coupled to the environment, for example, re-tracing the path of a graph while holding the chalk barely off the blackboard. Because our focus was on gestures that were not deictic, and these enactments involved coupling gesture with a preexisting blackboard inscription, no enactments were coded in this study. In depiction, the gesturer uses the index finger or some other physical element to outline or "draw" the shape of an object in the air. In the current setting, such "objects" consisted of mathematical entities, given a spatial representation in virtue of their definition, standard graphical inscription, or spatial conceptualization. For example, in Figure 10.3, the participant uses a sequence of staccato beat gestures to delimit a region in space, thus depicting this mathematical "object." Similarly, the second frame of Figure 10.4 shows a U-shaped gesture bounding a region "to the left." It should be noted that these gestural depictions of numerical intervals depend on a preestablished conceptual mapping between number and space, so that a "region" can be interpreted as a bounded portion of the real number line. This conceptual mapping is introduced early within mathematics teaching through representations such as the number line, in which numbers that are "less than" a reference number are conventionally taught as lying to the left of the reference number (Dehaene, Bossini, & Giraux, 1993; Edwards, 2009; Lakoff & Núñez, 2000; Núñez, 2011; Núñez & Marghetis, in press). Similarly, in the two examples of gestures co-produced with the lexical affiliate increasing, both participants exhibited a marked rightward hand trajectory, mirroring the contingent fact that, historically, graphs of functions have been drawn from left to right. In general, the process of associating number and space can be accounted for by the cognitive mechanisms of conceptual blending and metaphor (Núñez, 2011), and a similar conceptual association between number and space has been shown in the context of arithmetic reasoning (Knops, Thirion, Hubbard, Michel, & Dehaene, 2009; McCrink, Dehaene, & Dehaene-Lambertz, 2007; Núñez & Marghetis, in press; Pinhas & Fischer, 2008).

In Kendon's category of modeling, the gesturing body part stands in for an object. Both specimens of dynamic gesture shown in Figures 10.1 and 10.2 involve the modeling of a mathematical concept via a fictive trajector. In this case, the trajectory is the increase of the mathematical sequence under discussion, and fictive motion involves the motion of an imagined trajector along this trajectory. In Figure 10.1, the moving hands model the dynamic increase of the sequence, with the hands standing in for the fictively moving trajector. In Figure 10.2, the hand models the trajector that travels along the path of a function, passing through the function's increasing values. Note that the modeling of a trajector moving along the path of a graph is a different representation than enacting the action of drawing the graph in the first place; in modeling, the gesture stands in for a (fictive) moving trajector, whereas in enactment, the gesture reproduces, in part, the action of drawing the graph. For both these examples, extended fingers indicate the direction of the trajectory, constituting a kind of vector representation of the trajector's fictive motion—quite unlike any hand shape that might have been used to hold the chalk while tracing a graph on the blackboard.

Kendon's analysis of precisely how a gesture relates to its referent is thus helpful in characterizing how gestures that co-occur with mathematical speech can represent mathematical ideas in different ways. To examine whether this use of gesture to represent mathematical ideas was truly systematic and reliable, we next looked quantitatively at the co-occurrence of dynamic and static gestures with lexical items associated with dynamic and static concepts.

DYNAMIC CONCEPTS, DYNAMIC GESTURES

The participants produced a large number of gestures, the majority of which were deictic or "pointing" gestures, anchored to inscriptions on the blackboard. Deictic gestures appeared to play a number of roles, including maintaining attention during particularly complex deductions and directing the attention of a collaborator to a salient inscription. While others have studied the significance of deictic gestures for mathematical communication and learning (e.g., Alibali et al., 1999; Goldin-Meadow & Wagner, 2005; Goldin-Meadow et al., 2009), here we focus on participants' representational gestures. Every participant but one produced representational gestures that were co-timed with the dynamic and static lexical items of interest. A total of 166 of these co-timed representational gestures were coded, for a mean of 13.8 coded gestures per participant. Of these, the majority of the gestures were coded as dynamic (50.6%); fewer were coded as static (41.6%).

In line with the hypothesis that gesture can express the metaphorical and/or spatial content of abstract mathematical ideas, gesture dynamicity varied according to the co-occurring concept expressed in mathematical speech (Marghetis & Núñez, in press). Certain concepts were associated with a prevalence of dynamic or static gestures. For example, as shown in Figure 10.5, gestures co-produced with talk of "increase," "continuity," and "intersection" were more often dynamic; those co-produced with talk of "containment" and "closeness" were more often static.

To test the robustness of this association between the dynamicity of mathematical concept and the dynamicity of gesture, we examined those participants who produced gestures that co-occurred with these concepts and calculated the proportion of gestures that were dynamic. We focused on specific mathematical concepts that, based on previous theoretical analyses, were thought to have a construal that was primarily dynamic or primarily static: the dynamic notion of increase—thought to evoke fictive motion—and the static notions of closeness and containment (Lakoff & Núñez, 2000). A between-subjects analysis of variance found that gestures produced while speaking of "increase" were significantly more often dynamic than static [$F(1,10) = 28.90$, $p = .0003$], while participants produced a significantly higher proportion of static gestures while discussing "containment" [($F(1,12) = 6.75$, $p = .0232$] and "closeness" [$F(1,10) = 76.73$, $p < .0001$].

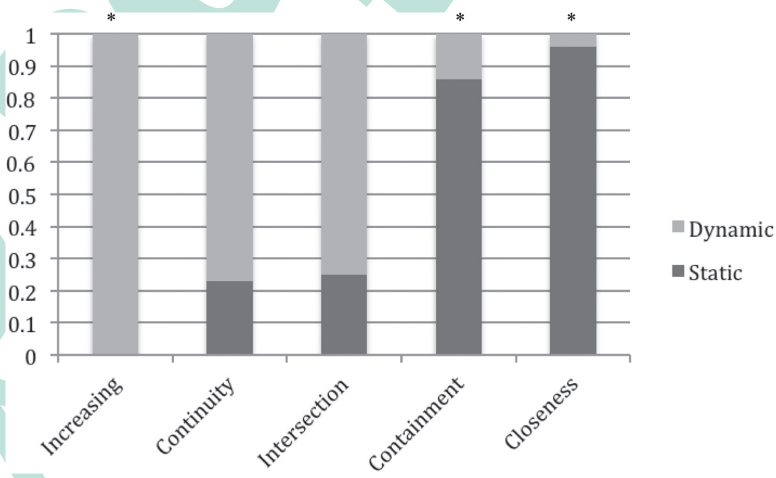

Figure 10.5 Proportion of dynamic and static gestures by co-occurring concept.

Among these expert practitioners, therefore, dynamicity in gesture was reliably associated with mathematical concepts thought to rely on dynamic conceptualization, thus lending credence to the hypothesis that even expert mathematical practice relies on intuitions of motion—a dynamic complement to the static definitions.

DISCUSSION

This empirical study of mathematical practice supports the claim that gesture is an integral component in the production and communication of mathematical meanings during expert proof, and furthermore that gestural meaning-making makes systematic use of dynamicity. Both dynamic and static representational gestures were widespread and recurrent in collaborative mathematical practice among doctoral students. Lexical items linked to mathematical concepts associated with dynamism or fictive motion were systematically paired with dynamic gestures, whereas terms associated with static concepts such as containment were systematically paired with static gestures. In summary, dynamic metaphorical gestures are not restricted to pedagogical or elementary mathematical contexts but are also clearly present in expert mathematical practice.

Because the construction of mathematical diagrams is necessarily a dynamic process, typically involving the tracing of chalk across a blackboard, one might wonder whether the dynamism of gesture in a context involving graphing is a mere echo of the dynamism of the inscriptive motion. Certainly, the motion that inevitably accompanies the creation of a mathematical diagram by hand is probably a factor in the historical and developmental origin of fictive motion for such notions as continuity, function, and limit. The contemporary dynamism of mathematical thought, however, is incredibly robust. Mathematical discourse is rife with fictive motion in the absence of diagrams altogether, as demonstrated by corpus studies of textbooks (Núñez et al., 1999; Núñez & Lakoff, 1998). As an example from the study analyzed here, in Figure 10.2, the participant produced a dynamic gesture not after creating a graphical inscription but after writing a series of static inequalities. The dynamism of his gesture reveals the dynamism of his understanding of increasing sequences. While a graph of a function is visible in Figure 10.1, the participants had not recently attended to the diagram; instead, they were discussing the limit of a function and representing it symbolically using set notation. Moreover, the participant's gesture moves orthogonally to the orientation of the blackboard diagram. The dynamism evinced in gesture is not merely an echo of inscriptive actions, therefore, but evidence for the dynamism inherent in the participants' understanding of the mathematics.

Attention to gesture, and to the body-in-interaction more generally, has opened up new vistas for further research on proof. In the current analysis, environmentally coupled deictic gestures were purposefully disregarded, as when participants pointed at inscriptions on the blackboard. Did this choice ignore an essential component of mathematical gestural behavior? Pointing at the graph of a function, for instance, was not coded, even if the hand dynamically swept along the curve of the graph. This aspect of the coding scheme represented a conscious decision to sacrifice breadth of analysis for experimental traction and focus. The coding scheme successfully discriminated between mathematical gestures that expressed mathematical content (e.g., "The function is increasing") and meta-mathematical gestures that conveyed something about the local practice and environment (e.g., "Let's move the diagram a bit higher on the blackboard"). While deictic gestures are undeniably an important part of mathematical practice, particularly when that practice involves collaboration or explanation at a blackboard (see Goldin-Meadow et al., 2009), the analysis of such environmentally coupled gestures requires care because their interpretation may require attention to the inscription, to the mathematical entity represented by that inscription, or to both (Edwards, 2009; Hutchins & Palen, 1998). By restricting our attention to representational gestures, the analysis avoided this complication. Future research will explore the ways in which environmentally coupled gestures elaborate meaning and direct local practices.

In attending exclusively to formal, disembodied theorems and proofs—products of the sedimentation of local mathematical practices—the study of mathematical thinking has ignored the rich meaning-making practices of flesh-and-blood mathematicians and collapsed multi-agent and multimodal practice into a single idealized agent working within a single written modality. In order to account for the exceptional traits of mathematics—objectivity, necessity, precision, stability—we must remember that, "Of course, in one sense, mathematics is a body of knowledge, but still it is also an *activity*" (Wittgenstein, 2009, para. 349).

In this chapter, we have faced off against the received wisdom about mathematics: that it is largely ahistorical, existing independently of human activity; that the real mathematics is the product of mathematical practice—text-based proofs, lemmas, and theorems—while the messy, fleshy activity of actually doing mathematical proof is secondary. Research that takes gesture and other bodily modalities into account is a direct response to this ahistorical tradition and a contribution to an emergent study of mathematics that does justice to actual mathematicians and mathematical activity (e.g., Mancosu, 2008). Without a doubt, expert mathematics is marked by rigorous methods, formal definitions, and symbol manipulation, but mathematical practice—and mathematical proof in particular—requires the mobilization of far more than these abstract resources. Contemporary expert proof, for instance, is a richly embodied practice that involves inscribing and manipulating notations, interacting with those notations through speech and gesture, and using the body to enact the meanings of mathematical ideas (Goldstone, Landy, & Son, 2010). By including gesture in the analysis, we take the mathematician's body seriously as a semiotic resource in the creation and communication of mathematical content. Seen in this light, mathematical proof is no longer an abstract product but is instead a dynamic practice, a human activity that involves talking, inscribing, and, crucially, gesturing: Mathematics is manual labor.

NOTES

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2. Because the goal of the study was to document the deployment of space to represent mathematical content, we excluded interactive gestures—used to manage the communication between the participants.

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